HW 7

1. (a) $\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}\,dz = 2\pi i\, e^{-\frac{\pi}{2} i} = 2\pi i(-i) = 2\pi$

(b) $\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i \frac{\cos 0}{0^2+8} = \frac{\pi i}{4}$

(c) $\int_C \frac{z\,dz}{2z+1}\,dz = \int_C \frac{1}{z+\frac{1}{2}}\,\frac{z}{2}\,dz = 2\pi i\left(\frac{-\frac{1}{2}}{2}\right) = -\frac{\pi i}{2}$

(d) $\int_C \frac{\cosh(z)}{z^4}\,dz = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\cosh(z)\Big|_{z=0} = \frac{2\pi i}{3!}\sinh(0) = 0$.

(e) $\int_C \frac{\tan(\frac{z}{2})}{(z-x_0)^2}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\tan(\frac{z}{2})\Big|_{z=x_0} = 2\pi i\cdot\frac{1}{2}\sec^2(\frac{z}{2})\Big|_{z=x_0} = \pi i\sec^2(\frac{x_0}{2})$

2. let C be the circle $|z-i|=2$

(a) $\int_C g(z)\,dz = \int_C \frac{1}{z-2i}\,\frac{1}{z+2i}\,dz = 2\pi i\frac{1}{2i+2i} = \frac{\pi}{2}$

(b) $\int_C g(z)\,dz = \int_C \frac{1}{(z-2i)^2}\cdot\frac{1}{(z+2i)^2}\,dz$

$= 2\pi i \frac{d}{dz}\frac{1}{(z+2i)^2}\Big|_{z=2i}$

$= 2\pi i\left(-\frac{2}{(z+2i)^3}\Big|_{z=2i}\right)$

$= 2\pi i\left(-\frac{2}{(4i)^3}\right)$

$= \frac{\pi}{16}$

3. $g(2) = \int_C \frac{2s^2-s-2}{s-2}\,ds$

$= 2\pi i(2\cdot 2^2-2-2)$, since 2 is inside C

$= 8\pi i$

For $|z|>3$, since $\frac{2s^2-s-2}{s-z}$ is analytic on and inside the circle, $g(z)=0$ by Cauchy Goursat thm.

④ $g(z) = \int_C \frac{s^3+2s}{(s-z)^3}\, dz$

$= \frac{2\pi i}{2!} \frac{d^2}{ds^2}(s^3+2s)\Big|_{s=z}$

$= \pi i \ [6s]_{s=z}$

$= 6\pi i z$

$g(z) = 0$ when z is outside by similar reason as Q3.

⑥. By Cauchy inequality, $\forall z_0 \in \mathbb{C}$,

$$|f^{(2)}(z_0)| \le \frac{2!}{R^2} \max_{z \in B_R(z_0)} |f(z)|$$

$$\le \frac{2}{R^2} \max_{z \in B_R(z_0)} |z|$$

$$\le \frac{2}{R^2} (|z_0| + R) \to 0 \text{ as } R \to \infty$$

So $f'(z)$ is a constant and $f(z) = a_1 z + a_0$.

Note that $|a_0| = |f(0)| \le 0 \Rightarrow a_0 = 0$

So $f(z) = a_1 z$.

⑤ By Cauchy integral formula,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i$$

We observed that

$$\int_C \frac{e^{az}}{z} dz = \int_0^{\pi} \frac{e^{az}}{e^{i\theta}} \cdot i e^{i\theta} d\theta + \int_{-\pi}^{0} \frac{e^{az}}{e^{i\theta}} \cdot i e^{i\theta} d\theta$$

Comparing the imaginary part,

$$\int_0^{\pi} e^{a\cos\theta} \cos(a\sin\theta)\, d\theta = \pi$$

⑦ If $f = u + iv$ and $g = e^f$, then

$$|g| = |e^f| = e^u \le e^{u_0} \quad \forall z \in \mathbb{C}.$$

Thus by Liouville thm, $g = e^f$ is a constant.

We see that $e^u \cos v$ and $e^u \sin v$ are constant.

By Cauchy Riemann equation, we see that u must be constant.

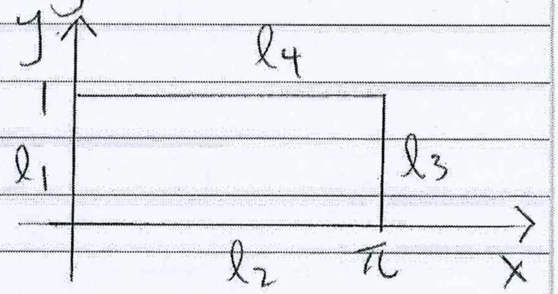
⑧ Since $f \neq 0$ in R and f is analytic in the interior of R, $1/f$ is analytic in the interior of R. By maximum modulus principle and the continuity of f in R, $1/|f|$ attains its maximum value on the boundary of R. Therefore, $|f|$ attains a minimum value m on the boundary of R and never in the interior.

⑨ Take R to be a closed unit disk and $f(0) = 0$.

⑩ By maximum modulus principle, the maximum value of $|f|$ attains on the boundary.

On l_1, $|f|^2 = \sinh^2 y$

$$\max_{l_1} |f|^2 = \frac{1}{4}(e - e^{-1})^2$$



⑪, ⑫ : Refer to Ex. of Tut 6.